

On the radical idealizer chain of symmetric orders.

Gabriele Nebe *

ABSTRACT: If Λ is an indecomposable, non-maximal, symmetric order, then the idealizer of the radical $\Gamma := \text{Id}(J(\Lambda)) = J(\Lambda)^\#$ is the dual of the radical. If Γ is hereditary then Λ has a Brauer tree (under modest additional assumptions). Otherwise $\Delta := \text{Id}(J(\Gamma)) = (J(\Gamma)^2)^\#$. If $\Lambda = \mathbb{Z}_p G$ for a p -group $G \neq 1$, then Γ is hereditary iff $G \cong C_p$ and otherwise $[\Delta : \Lambda] = p^2 |G/(G'G^p)|$. For Abelian groups G , the length of the radical idealizer chain of $\mathbb{Z}_p G$ is $(n-a)(p^a - p^{a-1}) + p^{a-1}$, where p^n is the order and p^a the exponent of the Sylow p -subgroup of G .

1 Introduction

Throughout this paper let R be a discrete valuation ring with maximal ideal πR and residue class field $R/\pi R =: k$. Let K be the field of fractions of R and \mathcal{A} a separable finite dimensional K -algebra. If Λ is an R -order in \mathcal{A} , then there is a canonical process, the so called **radical idealizer process**, that constructs an ascending chain of over-orders of Λ ending in a hereditary order Λ_N , called the **head order** of Λ (see Remark 2.7). We call the length of this chain the **radical idealizer length** $l_{rad}(\Lambda)$ of Λ . Hereditary orders are well understood, see [Jac], [Rei, Chapter 9]. They are direct sums of hereditary orders in the simple components of \mathcal{A} . So one might hope to classify R -orders according to the length of the radical idealizer chain and the head order.

An important class of R -orders are the **symmetric orders**, that are self dual with respect to some trace bilinear form on \mathcal{A} . Examples of symmetric orders are provided by blocks of group rings RG for finite groups G . The main tool to deal with symmetric orders is Jacobinski's conductor formula (see Theorem 4.3) stating that for any over-order Γ of a symmetric order Λ the conductor $F_\Gamma(\Lambda)$ (which is the largest Γ ideal in Λ) is the dual of Γ . If $\Gamma = \text{Id}(J(\Lambda))$ is the idealizer of $J(\Lambda)$, then a converse of this formula holds: Theorem 5.1 shows that for indecomposable, non-hereditary, symmetric orders Λ the dual of $J(\Lambda)$ is the idealizer of $J(\Lambda)$. Using his conductor formula, Jacobinski shows that the indecomposable symmetric orders Λ with $l_{rad}(\Lambda) = 0$ are maximal orders (Theorem 4.6). If $l_{rad}(\Lambda) = 1$, then one may derive the Brauer tree of Λ using the idealizer of $J(\Lambda)$ (see Proposition 7.2 and [Jac, Section 11]). In the present paper the first two steps of the radical idealizer chain for symmetric orders are investigated and properties of symmetric orders Λ with $l_{rad}(\Lambda) = 2$ are determined. We apply the theorems to p -groups G showing that $l_{rad}(\mathbb{Z}_p G) = 1$ if and only if $G \cong C_p$, $l_{rad}(\mathbb{Z}_p G) = 2$ if and only if $G \cong C_4$ or $G \cong C_2 \times C_2$. Moreover we calculate $l_{rad}(\mathbb{Z}_p G)$ for Abelian groups G .

*Lehrstuhl D für Mathematik, RWTH Aachen, 52056 Aachen, nebe@math.rwth-aachen.de

2 The radical idealizer chain

Let Λ be an R -order in \mathcal{A} . Then the **Jacobson radical** $J(\Lambda)$ is the intersection of all maximal right ideals of Λ . It is a 2-sided ideal of Λ , in fact the smallest ideal I of Λ , such that Λ/I is a semi-simple k -algebra. One other important characterization of $J(\Lambda)$ is that $J(\Lambda)$ is the biggest Λ -ideal I in Λ that is **pro-nilpotent**, i.e. for which there is $m \in \mathbb{N}$ such that $I^m \subset \pi\Lambda$ (cf. [Jac, Lemma 8.5]).

Definition 2.1. (see [Rei, Section 39]) Let Λ, Λ' be R -orders in \mathcal{A} . Then Λ **radically covers** Λ' , $\Lambda \succ \Lambda'$, if $\Lambda \supseteq \Lambda'$ and $J(\Lambda) \supseteq J(\Lambda')$. If Λ is maximal with respect to \succ , then Λ is called **extremal**.

Lemma 2.2. Let $\Gamma \succ \Lambda$ be two R -orders in \mathcal{A} . Then $J(\Lambda) = J(\Gamma) \cap \Lambda$ and $\Lambda/J(\Lambda)$ is isomorphic to a sub-algebra of $\Gamma/J(\Gamma)$. Moreover every simple Γ -module is semi-simple as a Λ -module.

Proof. Since $J(\Gamma) \cap \Lambda$ is an ideal of Λ that is nilpotent modulo $\pi\Lambda$, it is contained in $J(\Lambda)$. On the other hand $J(\Lambda) \subseteq J(\Gamma)$, because $\Gamma \succ \Lambda$. Therefore $J(\Gamma) \cap \Lambda = J(\Lambda)$ and $\Lambda/J(\Lambda) \cong (\Lambda + J(\Gamma))/J(\Gamma)$ is naturally embedded in $\Gamma/J(\Gamma)$. The second assertion follows from the fact that $\Gamma J(\Lambda) = J(\Lambda) \subseteq J(\Gamma)$ which implies that $\Gamma/J(\Gamma)$ is a semi-simple Λ -module. \square

Recall that an order Γ is called **hereditary**, if every left ideal of Γ is projective (see [Rei, Section 10]).

Theorem 2.3. ([Jac, Satz 8.12], [Rei, Theorem 39.14]) An R -order Λ in \mathcal{A} is **extremal**, if and only if Λ is **hereditary**.

Definition 2.4. Let L be a full R -lattice in \mathcal{A} . The **left order** of L is $O_l(L) := \{a \in \mathcal{A} \mid aL \subseteq L\}$. Analogously one defines the **right order** $O_r(L)$ of L . $\text{Id}(L) := O_l(L) \cap O_r(L)$ is called the **idealizer** of L .

Remark 2.5. Let Λ be an order and let Γ be one of $O_l(J(\Lambda))$, $O_r(J(\Lambda))$, or $\text{Id}(J(\Lambda))$. Then $\Gamma \succ \Lambda$.

The following characterization of hereditary orders is shown in [Rei, Theorem 39.11] for $O_l(J(\Lambda))$ instead of $\text{Id}(J(\Lambda))$. With a completely analogous proof (see [Neb]) one shows

Theorem 2.6. Let Λ be an R -order in \mathcal{A} . Then $\Lambda = \text{Id}(J(\Lambda))$ if and only if Λ is **hereditary**.

Remark 2.7. (cf. [BeZ]) Letting $\Lambda_0 := \Lambda$ and $\Lambda_{n+1} := \text{Id}(J(\Lambda_n))$ for $n = 0, 1, 2, \dots$ defines a canonical process, the so called **radical idealizer process** that constructs from an R -order Λ in \mathcal{A} successively bigger R -orders $\Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda_N = \Lambda_{N+1}$, the so called **radical idealizer chain**. The order Λ_N is **hereditary** and called the **head order** of Λ . If N is minimal such that $\Lambda_N = \Lambda_{N+1}$, then N is called the **radical idealizer length** $l_{\text{rad}}(\Lambda)$ of Λ .

Replacing Id by O_l respectively O_r , one can define left- and right-idealizer chain similarly. Since automorphisms of Λ preserve the radical, they also yield automorphisms of $O_r(J(\Lambda))$, $O_l(J(\Lambda))$ and $\text{Id}(J(\Lambda))$. The advantage of taking two-sided idealizers is that $\text{Id}(J(\Lambda))$ is also preserved under anti-automorphisms of Λ , which interchange $O_r(J(\Lambda))$ and $O_l(J(\Lambda))$ and hence left- and right-idealizer chains.

The next remark gives a lower bound on the length of the radical idealizer chain and is also useful for the explicit calculation of $\text{Id}(J(\Lambda))$, since one may calculate modulo the maximal ideal πR :

Remark 2.8. *Let Λ be an R -order in \mathcal{A} and $\Gamma := \text{Id}(J(\Lambda))$. Then $J(Z(\Lambda))\Gamma \subseteq \Lambda$, in particular $\pi\Gamma \subseteq \Lambda$.*

Proof. $J(Z(\Lambda))\Lambda \subset J(\Lambda) \subset \Lambda$, since $J(Z(\Lambda))\Lambda$ is nilpotent modulo $\pi\Lambda$. Therefore

$$J(Z(\Lambda))\Gamma = J(Z(\Lambda))\Lambda\Gamma \subseteq J(\Lambda)\Gamma = J(\Lambda) \subseteq \Lambda.$$

□

Definition 2.9. *Let Λ be an R -order in \mathcal{A} and let $\epsilon_1, \dots, \epsilon_s$ be the central primitive idempotents of \mathcal{A} . Then the defect of Λ is the minimal d such that $\pi^d \epsilon_t \in \Lambda$ for all $1 \leq t \leq s$.*

Note that this coincides with the usual definition of defect for blocks of group rings, if K is an unramified extension of \mathbb{Q}_p .

Since hereditary orders contain the central primitive idempotents of \mathcal{A} , one gets the following corollary:

Corollary 2.10. *The radical idealizer length $l_{\text{rad}}(\Lambda)$ is greater or equal than the defect of Λ .*

Definition 2.11. *For two R -orders Λ, Γ in \mathcal{A} , the conductor of Γ in Λ is the biggest 2-sided Γ -ideal $F_\Gamma(\Lambda)$ that is contained in Λ . Analogously one defines the left conductor $F_\Gamma^{(l)}(\Lambda)$ and the right conductor $F_\Gamma^{(r)}(\Lambda)$ as the largest left- respectively right-ideal of Γ contained in Λ .*

The following lemma is a straightforward generalization of [CPW, Theorem 2.2]:

Lemma 2.12. *Assume that \mathcal{A} is commutative, let $\epsilon_1, \dots, \epsilon_s$ be the primitive idempotents in \mathcal{A} and assume that $\Gamma := \bigoplus_{i=1}^s \epsilon_i \Lambda$ is the maximal order in \mathcal{A} . For $i \in \{1, \dots, s\}$ let π_i be a prime element in $\epsilon_i \Lambda$ and put $\pi := (\pi_1, \dots, \pi_s) \in \Gamma$. Let*

$$\Lambda = \Lambda_0 \subset \Lambda_1 \subset \dots \subset \Lambda_N = \Gamma$$

be the radical idealizer chain of Λ . Then for $n = 0, \dots, N$

$$F_\Gamma(\Lambda_n) = \pi^{-n} F_\Gamma(\Lambda) \cap \Gamma.$$

Proof. We argue by induction on n , the case $n = 0$ being trivial. Assume that

$$F_\Gamma(\Lambda_n) = \pi^{-n} F_\Gamma(\Lambda) \cap \Gamma = \bigoplus_{i=1}^s \pi_i^{a_i} \Lambda \epsilon_i.$$

Splitting off the direct summands of Λ_n that are maximal orders, we may assume that $a_i > 0$ for all i . Then

$$F_\Gamma(\Lambda_n) \subseteq J(\Lambda_n) = \pi \Gamma \cap \Lambda_n.$$

Since $F_\Gamma(\Lambda_n)$ is a Γ -ideal one gets

$$\pi^{-1} F_\Gamma(\Lambda_n) J(\Lambda_n) \subseteq \pi^{-1} F_\Gamma(\Lambda_n) \pi \Gamma \subseteq F_\Gamma(\Lambda_n) \subseteq J(\Lambda_n).$$

Hence

$$\pi^{-1} F_\Gamma(\Lambda_n) \subseteq \text{Id}(J(\Lambda_n)) = \Lambda_{n+1}$$

and therefore

$$F_\Gamma(\Lambda_{n+1}) \supseteq \pi^{-(n+1)} F_\Gamma(\Lambda) \cap \Gamma.$$

The opposite inclusion follows from Remark 2.8. □

Corollary 2.13. *In the notation of Lemma 2.12 let*

$$F_\Gamma(\Lambda) = \bigoplus_{i=1}^s \pi_i^{a_i} \Lambda \epsilon_i.$$

Then $l_{rad}(\Lambda) = \max_{i=1, \dots, s} a_i$.

3 Idealizers and bilinear forms

Idealizers can be calculated using a symmetric non-degenerate bilinear form

$$\phi : \mathcal{A} \times \mathcal{A} \rightarrow K \text{ that is associative, i.e. } \phi(ab, c) = \phi(a, bc) \text{ for all } a, b, c \in \mathcal{A}.$$

It is easy to see that such an associative bilinear form ϕ is of the form

$$\text{Tr}_z : \mathcal{A} \times \mathcal{A} \rightarrow K, (a, b) \mapsto \text{tr}_{red}(zab),$$

where $z \in Z(\mathcal{A})^*$ is an invertible element of the center of \mathcal{A} and tr_{red} denotes the reduced trace of \mathcal{A} . Fix such an associative symmetric bilinear form $\phi = \text{Tr}_z$. For a full R -lattice L in \mathcal{A} let

$$L^\# := \{a \in \mathcal{A} \mid \phi(L, a) \subset R\}$$

be the **dual lattice** with respect to ϕ . It is frequently used that dualizing is an inclusion reversing bijection of the set of full R -lattices in \mathcal{A} and that $(L^\#)^\# = L$ for all full R -lattices L in \mathcal{A} .

Lemma 3.1. *Let Γ be an R -order in \mathcal{A} . Then $\Gamma^\#$ is a 2-sided Γ -ideal with*

$$\Gamma = O_l(\Gamma^\#) = O_r(\Gamma^\#) = \text{Id}(\Gamma^\#).$$

Proof. Let $x, \gamma \in \Gamma$ and $y \in \Gamma^\#$. Then $\phi(x, \gamma y) = \phi(x\gamma, y) \in R$ and $\phi(y\gamma, x) = \phi(y, \gamma x) \in R$ and therefore $\Gamma \subset \text{Id}(\Gamma^\#)$. On the other hand let $\lambda \in O_l(\Gamma^\#)$. Then for all $y \in \Gamma^\#$ and $x \in \Gamma = (\Gamma^\#)^\#$

$$\phi(x\lambda, y) = \phi(x, \lambda y) \in R.$$

Hence $\Gamma\lambda \subseteq (\Gamma^\#)^\# = \Gamma$ and therefore $\lambda = 1\lambda \in \Gamma$. Analogously one gets $O_r(\Gamma^\#) \subseteq \Gamma$. \square

Proposition 3.2. *If L is a full R -lattice in \mathcal{A} , then $O_l(L) = (LL^\#)^\#$, $O_r(L) = (L^\#L)^\#$, and hence*

$$\text{Id}(L) = (LL^\#)^\# \cap (L^\#L)^\#.$$

Proof. We only show $O_l(L) = (LL^\#)^\#$. Let $\gamma \in \mathcal{A}$. Then $\gamma \in (LL^\#)^\#$ if and only if for all $x \in L$, $y \in L^\#$

$$\phi(\gamma, xy) = \phi(\gamma x, y) \in R$$

which is equivalent to $\gamma L \subseteq (L^\#)^\# = L$, i.e. $\gamma \in O_l(L)$. Analogously $O_r(L) = (L^\#L)^\#$. \square

From this one gets an interesting direct description of the idealizer of the radical.

Corollary 3.3. *Let Λ be an R -order in \mathcal{A} and $\Gamma := \text{Id}(J(\Lambda))$. Then Γ is the biggest Λ -ideal $I \subset \frac{1}{\pi}\Lambda$ such that $I/J(\Lambda)$ is a semi-simple Λ - Λ -bimodule.*

Proof. By Proposition 3.2

$$\Gamma^\# = (J(\Lambda)J(\Lambda)^\#) + (J(\Lambda)^\#J(\Lambda))$$

is the smallest Λ -ideal J in $J(\Lambda)^\#$ for which $J(\Lambda)^\# / J$ is a semi-simple Λ - Λ -bimodule. Since the dual of a bimodule is semi-simple if and only if the module is semi-simple, the corollary follows. \square

4 Symmetric orders.

Definition 4.1. *An R -order Λ in \mathcal{A} is called **symmetric**, if there is a non-degenerate, symmetric, associative, bilinear form $\phi = \text{Tr}_z : \mathcal{A} \times \mathcal{A} \rightarrow K$ with $\Lambda = \Lambda^\#$.*

Lemma 4.2. *(see e.g. [The, Proposition (1.6.2)]) Let $e, f \in \Lambda$ be two idempotents in the symmetric order Λ . Then $\phi|_{e\Lambda f \times f\Lambda e}$ is a regular R -bilinear pairing. In particular $e\Lambda e$ is a symmetric order in $e\mathcal{A}e$.*

An important tool to deal with symmetric orders is Jacobinski's conductor formula:

Theorem 4.3. *([Jac, Satz 10.6]) Let Λ be a symmetric R -order in \mathcal{A} and $\Gamma \supseteq \Lambda$ an over-order. Then left and right conductor coincide and are equal to the dual of Γ :*

$$F_\Gamma(\Lambda) = F_\Gamma^{(l)}(\Lambda) = F_\Gamma^{(r)}(\Lambda) = \Gamma^\#.$$

Proof. Since $\Gamma^\# \subseteq \Lambda$ is a Γ -ideal by Lemma 3.1 one has $\Gamma^\# \subseteq F_\Gamma(\Lambda)$. On the other hand if $x \in F_\Gamma(\Lambda)$ and $\gamma \in \Gamma$ then $\phi(x, \gamma) = \phi(x\gamma, 1) \in R$ is integral, since $x\gamma \in \Lambda = \Lambda^\#$. \square

From this proof one even gets that $\Gamma^\#$ is the largest R -lattice L in Λ with $\Gamma L \subseteq \Lambda$. [Jac, Satz 10.7] and [Ple, Theorem III.8] describe the conductor of $F_\Gamma(\Lambda)$ for hereditary and (more general) graduated over-orders Γ of the symmetric order Λ . To apply this precise version of the conductor formula, we need the following (technical) notation:

Notation 4.4. Let $\epsilon_1, \dots, \epsilon_s$ be the central primitive idempotents in \mathcal{A} .

Let $z \in Z(\mathcal{A})$ be such that $\Lambda = \Lambda^\#$ with respect to Tr_z and

$z_i = \epsilon_i z \in Z(\epsilon_i \mathcal{A}) = K_i$ ($1 \leq i \leq s$).

Let R_i be the maximal order in K_i with maximal ideal \wp_i ,

$\wp_i^{-d_i}$ the inverse different of R_i over R and $n_i \in \mathbb{Z}$ with $z_i R_i = \wp_i^{-n_i}$ ($1 \leq i \leq s$).

The simple algebra $\mathcal{A}\epsilon_i$ is isomorphic to a matrix ring over a central K_i -division algebra D_i . Let Ω_i be the maximal order in D_i and $m_i^2 = \dim_{K_i}(D_i)$.

Theorem 4.5. ([Jac, Satz 10.7], [Ple, Theorem III.8]) With the notation above let Δ be a hereditary order in \mathcal{A} . Then

$$\Delta^\# = \bigoplus_{i=1}^s \wp_i^{m_i(n_i-d_i-1)} J(\Delta) \epsilon_i.$$

Theorem 4.6. Let Λ be an indecomposable symmetric R -order in \mathcal{A} and $0 \neq e^2 = e \in \Lambda$ be an idempotent such that $e\Lambda e$ is hereditary. Then Λ is a maximal order.

Proof. $\epsilon_i e \Lambda e =: \Lambda_i$ is either $\{0\}$ or a symmetric hereditary order in $e\mathcal{A}e$ for all $1 \leq i \leq s$.

Let i be fixed such that $\Lambda_i \neq \{0\}$. Then the conductor formula 4.5 yields that

$$\Lambda_i = \Lambda_i^\# = \wp_i^{m_i(n_i-d_i-1)} J(\Lambda_i).$$

In particular Λ_i is isomorphic to $J(\Lambda_i)$ as a bimodule and therefore Λ_i a maximal order in $\epsilon_i e \mathcal{A} e$, $m_i = 1$ and $n_i = d_i$. But then the conductor of every maximal over-order Γ of Λ in Λ is of the form $F_\Gamma(\Lambda) = \Gamma' \oplus \epsilon_i \Gamma$ for a suitable order Γ' . In particular $\epsilon_i \in \Lambda$. Since Λ is indecomposable $\Lambda = \epsilon_i \Lambda = \epsilon_i \Gamma$ and Λ is a maximal order in the simple K -algebra $\mathcal{A} = \epsilon_i \mathcal{A}$. \square

Putting $e = 1$ in Theorem 4.6 this characterizes the symmetric orders Λ with $l_{rad}(\Lambda) = 0$ as maximal orders. In particular if Λ is a block of a group ring RG , then $l_{rad}(\Lambda) = 0$ if and only if the defect of Λ is 0 (see [Jac, Satz 11.1]).

5 The radical idealizer of symmetric orders.

In this and the next section the first two steps of the radical idealizer chain of symmetric orders are made precise. The first theorem is a sort of converse of the conductor formula.

Theorem 5.1. Let Λ be a non-hereditary, indecomposable, symmetric R -order in \mathcal{A} and $\Gamma := \text{Id}(J(\Lambda))$ the idealizer of the radical of Λ . Then $\Gamma = J(\Lambda)^\#$.

Proof. $\Gamma^\# \subseteq \Lambda$ is the largest Γ -ideal in Λ by the conductor formula. Since $J(\Lambda) \subseteq \Lambda$ is a Γ -ideal, one has $J(\Lambda) \subseteq \Gamma^\#$ and therefore $\Gamma \subseteq J(\Lambda)^\#$.

To show the converse inclusion let $e_1, \dots, e_h \in \Lambda$ be orthogonal idempotents that map onto the central primitive idempotents of $\Lambda/J(\Lambda)$ with $1 = e_1 + \dots + e_h$. Then $\Lambda = \bigoplus_{i,j=1}^h e_i \Lambda e_j$, $(e_i \Lambda e_j)^\# = e_j \Lambda e_i$ and $e_i \Lambda e_i$ is a symmetric R -order in $e_i \mathcal{A} e_i$, which is not hereditary because of Theorem 4.6. Now

$$J(\Lambda) = \bigoplus_{i \neq j=1}^h e_i \Lambda e_j \oplus \bigoplus_{i=1}^h J(e_i \Lambda e_i)$$

and with Lemma 4.2

$$J(\Lambda)^\# = \bigoplus_{i \neq j=1}^h e_i \Lambda e_j \oplus \bigoplus_{i=1}^h J(e_i \Lambda e_i)^\#.$$

Assume first that $h = 1$ so $\Lambda/J(\Lambda)$ is a simple k -algebra. Then $J(\Lambda)$ is a maximal 2-sided ideal in Λ . Therefore either $\Gamma^\# = \Lambda$ or $\Gamma^\# = J(\Lambda)$. In the first case $\Gamma = (\Gamma^\#)^\# = \Lambda^\# = \Lambda$ and therefore Λ is hereditary contradicting the assumption. In the second case $\Gamma = J(\Lambda)^\#$ and the theorem follows.

Now let h be arbitrary and $\Gamma_j := J(e_j \Lambda e_j)^\#$. From above $\Gamma_j = \text{Id}(J(e_j \Lambda e_j))$ is an order, so it remains to show that for $i \neq j$ the summand $e_i \Lambda e_j$ of $J(\Lambda)^\#$ is a Γ_i - Γ_j -bimodule. The inclusion $(e_i \Lambda e_j) \Gamma_j \subseteq e_i \Lambda e_j$ is equivalent to

$$e_j \Lambda e_i = (e_i \Lambda e_j)^\# \subseteq (e_i \Lambda e_j \Gamma_j)^\#.$$

So let $e_j a e_j \in \Gamma_j$ with $a \in J(\Lambda)^\#$ and $\gamma, \lambda \in \Lambda$. Then

$$\phi(e_i \lambda e_j e_j a e_j, e_j \gamma e_i) = \phi(e_j \gamma e_i, e_i \lambda e_j a e_j) = \phi(e_j \gamma e_i \lambda e_j, e_j a e_j) \in R,$$

because $e_j \gamma e_i \lambda e_j \in J(\Lambda)$ (note that $i \neq j$) and $a \in J(\Lambda)^\#$. Analogously $\Gamma_i(e_i \Lambda e_j) \subseteq e_i \Lambda e_j$.

Since $J(e_j \Lambda e_j)$ is a Γ_j -bimodule ($1 \leq j \leq s$), one gets $J(\Lambda)^\# J(\Lambda) J(\Lambda)^\# \subseteq J(\Lambda)$ and hence $J(\Lambda)^\# \subseteq \Gamma$. \square

Since $\text{Id}(J(\Lambda)) \subseteq O_r(J(\Lambda))$ and the right-conductor $O_r(J(\Lambda))^\# \supset J(\Lambda)$ one gets the same result for the left- and right-idealizer of $J(\Lambda)$.

Corollary 5.2. *Let Λ be a symmetric order. Then*

$$\text{Id}(J(\Lambda)) = O_r(J(\Lambda)) = O_l(J(\Lambda)).$$

The orders $e \Lambda e$, where $e^2 = e \in \Lambda$ is an idempotent in Λ mapping onto a central primitive idempotent of $\Lambda/J(\Lambda)$ play an important role in the above proof. These orders are 2-sided local orders, i.e. they have a unique maximal 2-sided ideal. These orders have a unique simple module, or equivalently $e \Lambda e / e J(\Lambda) e$ is a simple k -algebra.

Lemma 5.3. *Let Λ be a 2-sided local, symmetric R -order and $\Gamma := \text{Id}(J(\Lambda))$. Then either $\Gamma/J(\Gamma) \cong \Lambda/J(\Lambda)$ as Λ - Λ -bimodule and $J(\Gamma) = J(\Gamma)^\#$ or $J(\Gamma) = J(\Lambda)$ and Γ is hereditary.*

Proof. Since $\Lambda/J(\Lambda)$ is a simple Λ - Λ -bimodule, also its dual Γ/Λ is simple. Now $\Gamma \supseteq J(\Gamma) + \Lambda \supseteq \Lambda$ and $J(\Lambda) = J(\Gamma) \cap \Lambda$. Therefore either $\Gamma = J(\Gamma) + \Lambda \neq \Lambda$ and $\Gamma/J(\Gamma) \cong \Lambda/J(\Lambda)$ or $J(\Gamma) \subseteq \Lambda$ whence $J(\Gamma) = J(\Lambda)$. In the latter case $\Gamma = \text{Id}(J(\Gamma))$ is hereditary. In the first case Λ is not hereditary and $J(\Gamma)$ is the unique maximal 2-sided Γ -ideal in Γ . Since $J(\Lambda) \subset J(\Gamma) \subset J(\Lambda)^\# = \Gamma$ (by Theorem 5.1), one also has $J(\Lambda) \subset J(\Gamma)^\# \subset \Gamma$. Here all inclusions are proper. So $J(\Gamma)^\#$ is also a maximal 2-sided Γ -ideal in Γ and hence $J(\Gamma)^\# = J(\Gamma)$. \square

Following the lines of [Jac, 11.4] one gets the following remark:

Remark 5.4. *With the assumptions of Lemma 5.3 assume that $J(\Lambda) = J(\Gamma)$. Then Γ is hereditary and $\Gamma/J(\Gamma)$ has two (isomorphic) composition factors as a Λ - Λ -bimodule, namely the submodule $\Lambda/J(\Lambda)$ and its dual, the factor module $\Gamma/\Lambda = (\Lambda/J(\Lambda))^\#$.*

Notation 5.5. *We fix the following notation:*

Λ denotes an indecomposable non-hereditary symmetric R -order in \mathcal{A} ,

$\Gamma := \text{Id}(J(\Lambda)) = J(\Lambda)^\#$ the idealizer of the radical of Λ ,

$\epsilon_1, \dots, \epsilon_s$ are the central primitive idempotents of \mathcal{A} , and

$e_1, \dots, e_h \in \Lambda$ are orthogonal lifts of the central primitive idempotents of $\Lambda/J(\Lambda)$.

According to Lemma 5.3 we order the e_i such that $e_i \Lambda e_i / J(e_i \Lambda e_i) \cong e_i \Gamma e_i / J(e_i \Gamma e_i)$ for $1 \leq i \leq t \leq h$ and $J(e_i \Gamma e_i) = J(e_i \Lambda e_i)$ for $t < i \leq h$ and put

$$e := \sum_{i=1}^t e_i \text{ and } f := \sum_{i=t+1}^h e_i.$$

From Lemma 5.3 one now gets

Corollary 5.6. *For $1 \leq i \leq h$ the idempotent $e_i + J(\Gamma)$ is a central idempotent of $\Gamma/J(\Gamma)$. If $1 \leq i \leq t$, then $e_i + J(\Gamma)$ is a central primitive idempotent. If $t < i \leq h$ then $e_i \Gamma e_i$ is hereditary.*

6 The second step of the radical idealizer chain

We keep the Notation 5.5. Moreover let $\Delta := \text{Id}(J(\Gamma))$ and let Λ_N be the head order of Λ .

Theorem 6.1.

$$\Delta = (J(e \Gamma e)^2 + e \Gamma f \Gamma e)^\# \oplus f \Gamma f \oplus f \Gamma e \oplus e \Gamma f.$$

Proof. $\Delta = ((J(\Gamma)^\# J(\Gamma) + J(\Gamma) J(\Gamma)^\#)^\#$ by Proposition 3.2. Now

$$J(\Gamma)^\# = (e J(\Gamma) e)^\# \oplus f \Gamma f \oplus e \Gamma f \oplus f \Gamma e.$$

Lemma 5.3 says $(e J(\Gamma) e)^\# = e J(\Gamma) e$ and therefore

$$\begin{aligned} J(\Gamma)^\# J(\Gamma) &= (e J(\Gamma) e \oplus f \Gamma f \oplus e \Gamma f \oplus f \Gamma e) J(\Gamma) = \\ &= ((e J(\Gamma) e)^2 + e \Gamma f \Gamma e) \oplus (f J(\Gamma) f + f \Gamma e \Gamma f) \oplus (f \Gamma e) \oplus (e J(\Gamma) e \Gamma f + e \Gamma f J(\Gamma) f). \end{aligned}$$

Since

$$f\Gamma e\Gamma f = f\Lambda e\Lambda f \subseteq J(f\Lambda f) = J(f\Gamma f)$$

one gets

$$\Delta^\# = ((eJ(\Gamma)e)^2 + e\Lambda f\Lambda e) \oplus fJ(\Lambda)f \oplus f\Lambda e \oplus e\Lambda f$$

and therefore the theorem follows. \square

Proposition 6.2. *For the head order Λ_N one finds*

$$f\Lambda_N f = f\Gamma f, \quad e\Lambda_N f = e\Lambda f, \quad f\Lambda_N e = f\Lambda e$$

and $e + J(\Lambda_N)$ and $f + J(\Lambda_N)$ are central idempotents of $\Lambda_N/J(\Lambda_N)$.

Proof. Let $\Lambda_0 := \Lambda$ and $\Lambda_i := \text{Id}(J(\Lambda_{i-1}))$ ($1 \leq i \leq N$).

Using induction we show that $f\Lambda_i f = f\Gamma f$, $e\Lambda_i f = e\Lambda f$, $f\Lambda_i e = f\Lambda e$ and that $f + J(\Lambda_i)$ (hence also $e + J(\Lambda_i)$) lies in the center of $\Lambda_i/J(\Lambda_i)$ for all $1 \leq i \leq N$.

For $i = 1$ this is trivial. Now let $i \geq 1$ and assume that the statement is true for i .

Then $\Lambda_{i+1} = ((J(\Lambda_i)^\# J(\Lambda_i) + J(\Lambda_i)J(\Lambda_i)^\#)^\#$. By assumption

$$J(\Lambda_i) = J(e\Lambda_i e) \oplus J(f\Lambda f) \oplus e\Lambda f \oplus f\Lambda e$$

and therefore

$$J(\Lambda_i)^\# = J(e\Lambda_i e)^\# \oplus f\Gamma f \oplus e\Lambda f \oplus f\Lambda e.$$

Since Λ_i radically covers Λ_{i-1} , it holds that $J(\Lambda_{i-1}) \subseteq J(\Lambda_i)$. In particular $J(\Lambda_0) \subseteq J(\Lambda_i)$. Therefore $J(\Lambda_i)^\# \subseteq J(\Lambda_0)^\# = \Gamma$. One calculates $J(\Lambda_i)^\# J(\Lambda_i) + J(\Lambda_i)J(\Lambda_i)^\#$

$$= (J(e\Lambda_i e)^\# J(e\Lambda_i e) + J(e\Lambda_i e)J(e\Lambda_i e)^\# + e\Lambda f\Lambda e) \oplus J(f\Lambda f) \oplus e\Lambda f \oplus f\Lambda e.$$

After dualizing, one gets the desired form of Λ_{i+1} .

Let $\lambda \in \Lambda_{i+1}$. Then

$$f\lambda - \lambda f = f\lambda e - e\lambda f \in e\Lambda_{i+1}f \oplus f\Lambda_{i+1}e = e\Lambda f \oplus f\Lambda e \subseteq J(\Lambda) \subseteq J(\Lambda_{i+1}).$$

Therefore $f + J(\Lambda_{i+1}) \in Z(\Lambda_{i+1}/J(\Lambda_{i+1}))$ and hence also $e + J(\Lambda_{i+1}) = (1 - f) + J(\Lambda_{i+1})$ is central. \square

Corollary 6.3. *The conductor of Λ_N in Λ is*

$$\Lambda_N^\# = (e\Lambda_N e)^\# \oplus fJ(\Lambda)f \oplus e\Lambda f \oplus f\Lambda e.$$

Theorem 6.4. *Let Λ , $\Gamma = \text{Id}(J(\Lambda))$, $\Delta = \text{Id}(J(\Gamma))$ and f be as in Notation 5.5.*

Then $f = 0$ or $f = 1$.

If $f = 1$, then $J(\Gamma) = J(\Lambda)$ and $\Gamma = \Delta$ is hereditary.

If $f = 0$, then $\Gamma = J(\Gamma) + \Lambda$, $\Gamma/J(\Gamma) \cong \Lambda/J(\Lambda)$, $J(\Gamma) = J(\Gamma)^\#$ and $\Delta = (J(\Gamma)^2)^\#$.

Proof. With Notation 4.4 the conductor formula 4.5 gives

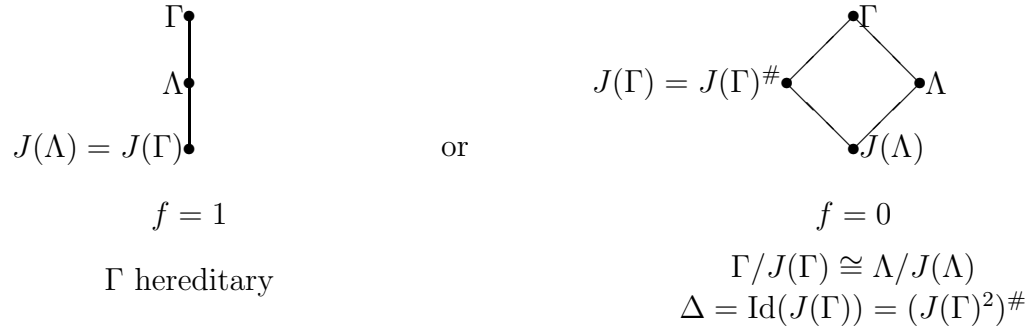
$$\Lambda_N^\# = (e\Lambda_N e)^\# \oplus fJ(\Lambda)f \oplus e\Lambda f \oplus f\Lambda e = \oplus_{i=1}^s \wp_i^{m_i(n_i-d_i-1)} (eJ(\Lambda_N)e \oplus fJ(\Lambda)f \oplus e\Lambda f \oplus f\Lambda e).$$

So if $\epsilon_i f \neq 0$ for some $1 \leq i \leq s$, then $n_i - d_i = 1$ and $J(\Lambda_N)\epsilon_i \subseteq \Lambda$. Since $J(\Lambda_N)\epsilon_i$ is a pro-nilpotent Λ -ideal in Λ containing $J(\Lambda)\epsilon_i$ one gets $J(\Lambda)\epsilon_i = J(\Lambda_N)\epsilon_i \subseteq J(\Lambda)$. But then $\epsilon_i \in \Gamma$ and $\Gamma\epsilon_i = \Lambda_N\epsilon_i$ is hereditary.

We claim that $f\epsilon_i = \epsilon_i$. To see this let $1 \leq j \leq t$ with $e_j\epsilon_i \neq 0$. Since $e_j + J(\Gamma)$ is a central primitive idempotent of $\Gamma/J(\Gamma)$, one even has $e_j\epsilon_i = e_j$. Now $\Gamma\epsilon_i$ is hereditary, so $e_j\Gamma e_j \cong \Omega_i^{x \times x}$ for some $x \in \mathbb{N}$. Let P_i denote the maximal ideal in Ω_i . Since $j \leq t$ Lemma 5.3 says that $P_i^{x \times x} \cong J(e_j\Gamma e_j)$ is symmetric with respect to the restriction of the form Tr_z above. But $n_i - d_i - 1 = 0$, yields together with [Rei, Theorem 14.9] that $J(e_j\Gamma e_j)^\# = \wp_i^{-1}e_j\Gamma e_j$ which is a contradiction. Therefore $e\epsilon_i = 0$ and hence $f\epsilon_i = \epsilon_i$.

So for all central primitive idempotents ϵ_i of \mathcal{A} either $f\epsilon_i = 0$ or $f\epsilon_i = \epsilon_i$. Therefore $\Lambda = e\Lambda e \oplus f\Lambda f$. Since Λ is assumed to be indecomposable one has $f = 0$ or $f = 1$. In the latter case $J(\Gamma) = J(\Lambda) = J(\Lambda_N)$, hence Γ is hereditary. If $f = 0$, then $\Delta = (J(\Gamma)^2)^\#$ from Theorem 6.1. The fact that $J(\Gamma)$ is self-dual follows with Lemma 5.3 which also implies that $\Gamma = J(\Gamma) + \Lambda$ and hence $\Lambda/J(\Lambda) \cong \Gamma/J(\Gamma)$. \square

Summarizing let Λ be an indecomposable, non-hereditary, symmetric R -order in \mathcal{A} , $\Gamma = \text{Id}(J(\Lambda))$ and $\Delta = \text{Id}(J(\Gamma))$. Let f be as in Notation 5.5. Then



Returning to the proof of Theorem 6.1 with the two possibilities $f = 1$ or $f = 0$ we find that either $\Gamma = \Delta = O_r(J(\Lambda))$ is hereditary or $J(\Gamma) = J(\Gamma)^\#$ whence $O_r(J(\Gamma)) = (J(\Gamma)^\# J(\Gamma))^\# = \Delta$. Therefore the first two steps in the right- and left-radical idealiser chain of a symmetric order Λ coincide.

Corollary 6.5. *Let Λ be a symmetric order, $\Gamma = \text{Id}(J(\Lambda))$ (which equals $O_r(J(\Lambda)) = O_l(J(\Lambda))$ by Corollary 5.2). Then*

$$\text{Id}(J(\Gamma)) = O_l(J(\Gamma)) = O_r(J(\Gamma)).$$

7 Symmetric orders with radical idealizer length 1 or 2.

The case $f = 1$ in Theorem 6.4 can be dealt with the arguments in [Jac, Satz 11.4]. With a modest additional assumption one gets $J(\Gamma) = J(\Lambda)$ and in particular if Λ is a block of a group ring with $l_{rad}(\Lambda) = 1$ one can associate a Brauer tree to Λ .

Proposition 7.1. *Assume that $\Gamma = \text{Id}(J(\Lambda))$ is hereditary. Assume further that $Z_i := \epsilon_i Z(\Lambda) \subseteq \epsilon_i \mathcal{A}$ is a maximal order for all central primitive idempotents $\epsilon_1, \dots, \epsilon_s$. Then*

$$J(\Gamma) = J(\Lambda).$$

Proof. With the Notation 4.4 and Jacobinski's conductor formula 4.5

$$J(\Lambda) = \Gamma^\# = \bigoplus_{i=1}^s \epsilon_i \mathcal{O}_i^{m_i(n_i - \delta_i - 1)} J(\Gamma).$$

By Remark 2.8 and since $\epsilon_i J(\Lambda) \subset J(\Lambda)$

$$J(Z_i)J(\Gamma) \subsetneq J(Z_i)\Gamma \subseteq J(\Lambda)$$

for all i . This implies that $n_i - \delta_i = 1$ and that $J(\Lambda) = J(\Gamma)$. \square

Proposition 7.2. *Assume that $J(\Gamma) = J(\Lambda)$. If moreover the decomposition map from the Grothendieck groups of simple modules $G_0(\mathcal{A}) \rightarrow G_0(k \otimes \Lambda)$ is surjective or k is a splitting field for $k \otimes \Gamma$, then for each idempotent $e_i \in \Lambda$, there are exactly two central primitive idempotents ϵ_{i_1} and ϵ_{i_2} in \mathcal{A} with $e_i \epsilon_{i_j} \neq 0$ ($j = 1, 2$).*

Proof. We make precise the embedding $\Lambda/J(\Lambda) \hookrightarrow \Gamma/J(\Gamma)$ following the lines of the proof of [Jac, Satz 11.4]: Let $e_i(\Lambda/J(\Lambda)) =: S_i$ be the simple algebra summand of $\Lambda/J(\Lambda)$ that corresponds to e_i ($1 \leq i \leq h$). Then

$$\Lambda/J(\Lambda) \cong S_1 \oplus \dots \oplus S_h \text{ and } \Gamma/\Lambda = (\Lambda/J(\Lambda))^\# \cong S_1^* \oplus \dots \oplus S_h^* \cong S_1 \oplus \dots \oplus S_h$$

as $\Lambda - \Lambda$ -bimodules. Therefore for every simple k -algebra summand S_i of $\Lambda/J(\Lambda)$ there are either two algebra-summands $T_{i'_1}$ and $T_{i'_2}$ of $\Gamma/J(\Gamma)$ such that S_i is diagonally embedded into $T_{i'_1} \oplus T_{i'_2}$ or there is a unique summand $T_i \cong l_2^{m \times n}$ such that $S_i \cong l_1^{n \times n} \subset T_i$ for extension fields l_2, l_1 of k with $[l_2 : l_1] = 2$. The latter is impossible, if k is a splitting field for $k \otimes \Gamma$. Similarly, if the decomposition map of Λ is surjective, the last case cannot happen, since otherwise the simple S_i -module occurs with even multiplicity in every simple Γ -module and hence in the reduction of every Γ -lattice modulo π . Therefore

$$S_i \hookrightarrow T_{i_1} \oplus T_{i_2} \subset \Gamma/J(\Gamma).$$

Since Γ is hereditary, it contains the central primitive idempotents ϵ_i of \mathcal{A} . Therefore $\Gamma/J(\Gamma) = \bigoplus_{i=1}^s \epsilon_i \Gamma/J(\epsilon_i \Gamma)$ and hence each simple summand of $\Gamma/J(\Gamma)$ is a summand of some $\epsilon_i \Gamma/J(\epsilon_i \Gamma)$. In particular for $j = 1, 2$ the summand $T_{i'_j}$ defines a unique central primitive idempotent ϵ_{i_j} with $f_{i'_j} \epsilon_{i_j} \neq 0$ for any lift $f_{i'_j} \in \Gamma$ of the central primitive idempotent of $\Gamma/J(\Gamma)$ that belongs to $T_{i'_j}$. Then ϵ_{i_j} ($j = 1, 2$) are the only central primitive idempotents in \mathcal{A} with $e_i \epsilon_{i_j} \neq 0$. \square

If the decomposition map of Λ is surjective (which is always satisfied when Λ is a block of a group ring) or k is a splitting field for $k \otimes \Gamma$ and the other assumptions of Proposition 7.2 hold, we can define a graph $\mathcal{G}(\Lambda)$ whose vertices correspond to $\epsilon_1, \dots, \epsilon_s$ and whose edges correspond to e_1, \dots, e_h . Two vertices ϵ_i and ϵ_j are connected by the edge e_l , if $e_l \epsilon_i \neq 0$ and $e_l \epsilon_j \neq 0$.

As in [Jac, Korollar 11.6] one shows:

Corollary 7.3. *If the decomposition map of Λ is surjective then $\mathcal{G}(\Lambda)$ is a tree.*

We end this section with a short remark on the length 2 case.

Remark 7.4. *With Notation 5.5 assume that $f = 0$ and $\Delta := \text{Id}(J(\Gamma))$ is hereditary.*

(i) $J(\Lambda)^2 \epsilon_i \in \Lambda$ for all $1 \leq i \leq s$. In particular $\pi^2 \epsilon_i \in \Lambda$ which means that the defect of Λ is ≤ 2 .

(ii) $\epsilon_i e_l \Lambda e_j \subseteq e_l \Lambda e_j$ for all $1 \leq i \leq s, 1 \leq l \neq j \leq h$.

(iii) If $s > 1$ then $\epsilon_i J(\Lambda) \not\subseteq J(\Lambda)$ for all $1 \leq i \leq s$.

Proof. (i) Let $1 \leq i \leq s$. Since $\epsilon_i \in \Delta$ one has $\epsilon_i J(\Gamma) \subseteq J(\Gamma)$. Now

$$J(\Lambda)^2 \subseteq J(\Gamma)^2 = \Delta^\# = \bigoplus_{i=1}^s J(\Gamma)^2 \epsilon_i \subseteq \Lambda$$

implies $J(\Lambda)^2 \epsilon_i \in \Lambda$. Since $\pi = \pi 1 \in \pi \Lambda \subset J(\Lambda)$ one has $\pi^2 \epsilon_i \in \Lambda$ for all i .

(ii) Let $l \neq j \in \{1, \dots, h\}$. Then $e_l \Lambda e_j \subseteq J(\Gamma)$. Since $\epsilon_i J(\Gamma) \subseteq J(\Gamma)$, the claim follows.

(iii) Assume that there is $1 \leq i \leq s$ such that $\epsilon_i J(\Lambda) \subseteq J(\Lambda)$. Then $\epsilon_i \in \Gamma$. Since $\epsilon_i \neq 1$ and Λ is indecomposable there is $1 \leq j \leq h$ such that $0 \neq \epsilon_i e_j \neq e_j$. But then $e_j + J(\Gamma) \in Z(\Gamma/J(\Gamma))$ is not primitive contradicting Corollary 5.6. \square

8 p -groups

Let $G \neq \{1\}$ be a p -group, $R = \mathbb{Z}_p$, and $\Lambda := RG$. Then Λ is a symmetric R -order with respect to the associative bilinear form

$$\phi(x, y) := \frac{1}{|G|} \text{trace}_{\text{reg}}(xy) = (xy)_1 \text{ if } xy = \sum_{g \in G} (xy)_g g$$

where $\text{trace}_{\text{reg}}$ is the regular trace of $\mathbb{Q}_p G$. Moreover

$$J(\Lambda) = \langle p\Lambda, g - h \mid g, h \in G \rangle_R$$

and

$$\Gamma := \text{Id}(J(\Lambda)) = J(\Lambda)^\# = \langle \Lambda, \frac{1}{p} \sum_{g \in G} g \rangle_R.$$

because $\frac{1}{p} \sum_{g \in G} g$ idealizes $J(\Lambda)$ and Γ is an over-order of Λ of index $p = |\Lambda/J(\Lambda)|$. If $|G| = p$ then Γ is hereditary by [Jac, Section 11]. Therefore we assume that $|G| \geq p^2$. Then the radical of Γ is

$$J(\Gamma) = \langle J(\Lambda), \frac{1}{p} \sum_{g \in G} g \rangle_R = J(\Gamma)^\#$$

and is contained in Γ of index p .

Theorem 8.1. *Let $\Delta := \text{Id}(J(\Gamma))$. Then*

$$\Delta = (J(\Gamma)^2)^\# = \langle \Lambda, \frac{1}{p^2} \sum_{g \in G} g, \frac{1}{p} \sum_{g \in G} \varphi(g)g \mid \varphi \in \text{Hom}(G, R/pR) \rangle_R.$$

Proof. Clearly $\lambda := \frac{1}{p^2} \sum_{g \in G} g \in \text{Id}(J(\Gamma))$. Let $y := \sum_{g \in G} y_g g \in \text{Id}(J(\Gamma))$. Then

$$p \sum_{g \in G} y_g g = a \frac{1}{p} \sum_{g \in G} g + x$$

with $x \in J(\Lambda) \subset \Lambda$ and $a \in R$. Hence $py_g \equiv \frac{a}{p} \pmod{R}$. Replacing y by $y - a\lambda$ we may assume that $y_g = \frac{a_g}{p}$ with $a_g \in R$ for all $g \in G$. Adding a suitable multiple of $p\lambda$ to y we can also assume that $a_1 = 0$. Now $\text{Id}(J(\Gamma)) = (J(\Gamma)^2)^\#$ and one calculates

$$J(\Gamma)^2 = \langle p^2\Lambda, p(g-h), (g_1-h_1)(g_2-h_2), \sum_{g \in G} g \mid g, h, g_1, h_1, g_2, h_2 \in G \rangle_R.$$

In particular the coefficient of 1 of $y(g^{-1}-1)(1-h^{-1})$ which is $\frac{1}{p}(a_g + a_h - a_{hg})$ lies in R . Hence

$$\varphi : g \mapsto a_g + pR \in R/pR$$

is a group homomorphism from G to R/pR , from which the inclusion \subseteq follows. It remains to show that the elements $y := \frac{1}{p} \sum_{g \in G} \varphi(g)g$ with $\varphi \in \text{Hom}(G, R/pR)$ are in the dual of $J(\Gamma)^2$. Clearly

$$\phi(y, p^2\Lambda) \subset R \text{ and } \phi(y, p(g-h)) \subset R \text{ for all } g, h \in G.$$

For $(g_1-h_1)(g_2-h_2)$ with $g_1, g_2, h_1, h_2 \in G$ one gets

$$\phi(y, (g_1-h_1)(g_2-h_2)) = \frac{1}{p}(\varphi(g_2^{-1}g_1^{-1}) - \varphi(h_2^{-1}g_1^{-1}) - \varphi(g_2^{-1}h_1^{-1}) + \varphi(h_2^{-1}h_1^{-1})) \in R$$

since φ is a homomorphism. The last generator is $p^2\lambda$ for which one finds

$$\phi(y, \sum_{g \in G} g) = \frac{1}{p} \sum_{g \in G} \varphi(g) \in R.$$

□

Corollary 8.2. *Let $\Lambda := \mathbb{Z}_p G$ for some p -group G of order $|G| \geq p^2$. Let $\Gamma := \text{Id}(J(\Lambda))$ and $\Delta := \text{Id}(J(\Gamma))$. Then $|\Gamma/\Lambda| = p$ and $|\Delta/\Lambda| = p^2|G/(G'G^p)|$.*

Corollary 8.3. *Let G be a p -group and $\Lambda := \mathbb{Z}_p G$.*

- 1) $\Gamma = \text{Id}(J(\Lambda))$ is hereditary if and only if $G \cong C_p$.
- 2) Assume that $|G| \geq p^2$. Then $\Delta = \text{Id}(J(\Gamma))$ is hereditary if and only if $|G| = 4$.

Proof. 1) Since $\Gamma \subset \frac{1}{p}\Lambda$ contains all central primitive idempotents of $\mathbb{Q}_p G$, one gets $|G| = p$. That $\text{Id}(J(\mathbb{Z}_p C_p))$ is hereditary follows from [Jac, Abschnitt 11].
 2) Since $\Delta \subset \frac{1}{p^2}\Lambda$ contains all central primitive idempotents of $\mathbb{Q}_p G$, one gets $|G| = p^2$. Hence G is Abelian, $G \cong C_{p^2}$ or $G \cong C_p \times C_p$ and by Corollary 8.2, $[\Delta : \Lambda] = p^3$ respectively p^4 . If Δ is hereditary, then Δ is the maximal order in $\mathbb{Q}_p G$,

$$\Delta \cong \mathbb{Z}_p \oplus \mathbb{Z}_p[\zeta_p] \oplus \mathbb{Z}_p[\zeta_{p^2}] \text{ respectively } \Delta \cong \mathbb{Z}_p \oplus \mathbb{Z}_p[\zeta_p]^{p+1}.$$

The form ϕ above is $\frac{1}{|\Gamma|} \text{trace}_{\text{reg}}$ hence the discriminant of Δ with respect to ϕ is

$$p^{-2p^2} \cdot 1 \cdot p^{p-2} \cdot p^{p(2p-3)} = p^{-2p-2} \text{ respectively } p^{-2p^2} \cdot 1 \cdot (p^{p-2})^{p+1} = p^{-p^2-p-2}$$

(see [Was, Prop. 2.1]). Since Δ contains a symmetric order of index p^3 respectively p^4 , one gets $-2p-2 = -6$ hence $p = 2$ respectively $-p^2-p-2 = 8$ which also implies $p = 2$. The same argument shows that for $G = C_2 \times C_2$ or $G = C_4$, the order Δ has the same discriminant as the maximal order and hence is hereditary (i.e. equal to the maximal order). \square

Note that this corollary also follows from Theorem 8.4 below.

For Abelian groups G , the radical idealizer length of $\mathbb{Z}_p G$ can be calculated from the exponent and the order of the Sylow p -subgroup of G :

Theorem 8.4. *Let G be an Abelian group with Sylow p -subgroup of order p^n and of exponent $p^a > 1$. Then*

$$l_{\text{rad}}(\mathbb{Z}_p G) = p^{a-1} + (p^a - p^{a-1})(n - a).$$

Proof. The theorem follows with Corollary 2.13 by calculating the conductor of the maximal order Γ in $\mathbb{Q}_p G$:

$$\Gamma = \bigoplus_{i=1}^s \mathbb{Z}_p G \epsilon_i = \bigoplus_{i=1}^s R_i[\zeta_{p^{a_i}}]$$

where R_i is an unramified extension of \mathbb{Z}_p and $\{a_1, \dots, a_s\} = \{0, 1, \dots, a\}$. If $*$ denotes the different, i.e. the dual with respect to the usual trace bilinear form, then by [Was, Prop. 2.1] $R_i[\zeta_{p^{a_i}}]^* = R_i[\zeta_{p^{a_i}}](1 - \zeta_{p^{a_i}})^{-p^{a_i-1}(a_i p - a_i - 1)}$ and hence the conductor of Γ in $\mathbb{Z}_p G$ is

$$\Gamma^\# = \bigoplus_{i=1}^s R_i[\zeta_{p^{a_i}}]^* p^n = \bigoplus_{i=1}^s R_i[\zeta_{p^{a_i}}](1 - \zeta_p^{a_i})^{((n-a_i)(p-1)+1)p^{a_i-1}}.$$

By Corollary 2.13, the length of the radical idealizer chain is

$$\max_{i=0, \dots, a} ((n-i)(p^i - p^{i-1}) + p^{i-1}) = (n-a)(p^a - p^{a-1}) + p^{a-1}.$$

\square

References

- [BeZ] H. Benz, H. Zassenhaus, *Über verschränkte Produktordnungen*. J. Number Theory **20** (1985), 282-298.
- [CPW] G. Cliff, W. Plesken, A. Weiss, *Order-Theoretic Properties of the Center of a Block*. Proceedings of Symposia in Pure Mathematics, AMS, 47 (1987), 413-420
- [Jac] H. Jacobinski, *Maximalordnungen und erbliche Ordnungen*. Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, Heft 6 (1981)
- [Neb] G. Nebe, *Orthogonale Darstellungen endlicher Gruppen und Gruppenringe*. Habilitationsschrift, RWTH Aachen, Aachener Beiträge zur Mathematik 26, Verlag Mainz, Aachen (1999)
- [Ple] W. Plesken, *Group rings of finite groups over the p -adic integers*. Springer Lecture Notes in Mathematics 1026 (1983).
- [Rei] I. Reiner, *Maximal Orders*. Academic Press, 1975.
- [The] J. Thévenaz, *G -Algebras and Modular Representation Theory*. Oxford Science Publications (1995)
- [Was] L.C. Washington, *Introduction to cyclotomic fields*. Springer Graduate Texts in Mathematics **83** (1982).